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## Weak-strong uniqueness for the Navier–Stokes–Poisson equations

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## ABSTRACT

This paper is devoted to study the weak-strong uniqueness property of compressible Navier–Stokes–Poisson system. By means of relative entropy method, we prove the result that the weak solution coincides with the strong solution, emanating from the same initial data, as long as the latter exists.

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## 1. Introduction

Let  $T > 0$  and  $\Omega = \mathbb{T}^3$  be the 3-dimensional torus. In this paper, we consider the following compressible Navier–Stokes–Poisson (NSP) system

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \quad (1.1)$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla p(\rho) = \rho \nabla \Phi + \rho f \quad (1.2)$$

$$\Delta \Phi = 4\pi g \left( \rho - \frac{1}{|\Omega|} \int_{\Omega} \rho dx \right) \quad (1.3)$$

in  $(0, T) \times \Omega$ , where  $\rho \in R$ ,  $\mathbf{u} \in R^3$  and  $\Phi \in R$  denote the electron density, electron velocity, and the electrostatic potential, respectively. The pressure function  $p(\rho)$  satisfies  $p(\rho) = a\rho^\gamma$  with  $a > 0$  and  $\gamma > 1$ .  $\mu, \lambda$  are the constant viscosity coefficients satisfying the physical requirements  $\mu > 0$ ,  $\lambda + \frac{2}{3}\mu \geq 0$ .  $f$  is the external force. The initial conditions are imposed as follows:

$$\rho(0, \cdot) = \rho_0(\cdot), \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0(\cdot), \quad (\rho \mathbf{u})(0, \cdot) = (\rho_0 \mathbf{u}_0)(\cdot) \quad (1.4)$$

The Navier–Stokes–Poisson system (1.1)–(1.3) can be used to describe the transportation of charge particles in electronic devices. More details about its background are introduced in [1]. Many researchers

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have been devoted to many topics of the compressible NSP system. Zhang and Tan [2] established the local existence of unique strong solution to the isentropic compressible Navier–Stokes–Poisson system (1.1)–(1.3) by means of Schauder fixed point theorem. The existence and asymptotic behavior of global solution are established by Li et al. [3] and Shi et al. [4]. The global well-posedness for NSP system in some Besov spaces has been investigated recently (see [5,6]). T. Kobayashi [7] proved the existence of finite-energy weak solutions of isentropic compressible NSP equation with the pressure  $p(\rho) = \rho^\gamma (\gamma > \frac{3}{2})$ . By use of Orlicz space theory, Zhang and Tan [8] got the existence of finite-energy weak solutions to NSP system with the pressure function  $p(\rho) = \rho \log^d \rho$  ( $d > 1$ ). Other properties of NSP system, for example, time delay rate and long-time behavior of solution, are investigated (see [9,10]). However, to best of our knowledge, there are no results about the weak–strong uniqueness of Navier–Stokes–Poisson equations. In this paper, we consider the weak–strong uniqueness principle of NSP system (1.1)–(1.3) which means that the weak solution must coincide with a strong solution emanating from the same initial data as long as the latter exists.

The relative entropy method is an important method to study partial differential equations. Carrillo et al. [11] used entropy dissipation method to consider the large-time asymptotic of quasilinear degenerate parabolic problems and proved the generalized Sobolev-inequalities. The relative entropy method is devoted to study the incompressible Euler limit of the Boltzmann equation in [12]. J. Glesselmann et al. [13] applied the modified relative entropy approach to derive the weak–strong stability of Navier–Stokes–Korteweg system. The weak–strong uniqueness property of the Navier–Stokes–Fourier system in bounded domain or unbounded domain is proved by E. Feireisl in [14] and Jessle et al. in [15]. In particular, the relative entropy method is the most important method to research weak–strong uniqueness of compressible Navier–Stokes equation with monotone pressure in [16–18] and non-monotone pressure in [19]. In [20], Kwon applied the refined relative entropy method to prove the convergence of the weak solution of degenerate compressible quantum NSP system to the strong solution of the incompressible Euler equation. Unfortunately, Kwon did not mention weak–strong uniqueness of Navier–Stokes–Poisson equations in [20]. In this paper, by the relative entropy method, we shall show weak–strong uniqueness property of Navier–Stokes–Poisson system for the first time.

The paper is organized as follows. In Section 2, we recall the definition of finite-energy weak solution for NSP equations (1.1)–(1.3) and state the main results. In Section 3, we derive the relative entropy inequality to (1.1)–(1.3). In Section 4, we prove the weak–strong uniqueness property of (1.1)–(1.3).

## 2. Main results

**Definition 2.1.** We call that  $(\rho, \mathbf{u}, \Phi)$  is a finite-energy weak solution to the Navier–Stokes–Poisson system (1.1)–(1.4) if

$$(i) \quad \rho \geq 0, \quad \rho \in L^\infty([0, T]; L^\gamma(\Omega)), \quad \mathbf{u} \in L^2([0, T]; W^{1,2}(\Omega)), \quad \Phi \in L^\infty([0, T]; W^{2,\gamma}(\Omega)). \quad (2.1)$$

(ii) The energy inequality

$$\frac{dE(t)}{dt} + \mu \int_{\Omega} |\nabla \mathbf{u}|^2 dx + (\lambda + \mu) \int_{\Omega} |\operatorname{div} \mathbf{u}|^2 dx \leq \int_{\Omega} \rho f \cdot \mathbf{u} dx \quad (2.2)$$

holds in  $\mathfrak{D}'((0, T))$  with the energy

$$E(t) = \int_{\Omega} \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{a}{\gamma - 1} \rho^\gamma + \frac{1}{8\pi g} |\nabla \Phi|^2 \right) (t, \cdot) dx < \infty \quad (2.3)$$

for  $t \in [0, T)$ .

(iii) For any  $\tau \in (0, T)$  and any test function  $\phi \in C^\infty([0, T] \times \Omega)$ , it holds

$$\int_{\Omega} \rho(\tau, \cdot) \phi(\tau, \cdot) dx - \int_{\Omega} \rho_0 \phi(0, \cdot) dx = \int_0^\tau \int_{\Omega} \rho \partial_t \phi + \rho \mathbf{u} \cdot \nabla \phi dx dt. \quad (2.4)$$

(iv) For any  $\tau \in (0, T)$  and any test function  $\varphi \in C^\infty([0, T] \times \Omega)$ ,  $\varphi|_{\partial\Omega} = 0$ , it holds

$$\begin{aligned} & \int_{\Omega} \rho \mathbf{u}(\tau, \cdot) \varphi(\tau, \cdot) dx - \int_{\Omega} \rho_0 \mathbf{u}_0 \varphi(0, \cdot) dx \\ &= \int_0^\tau \int_{\Omega} \rho \mathbf{u} \partial_t \varphi + (\rho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi - \mu \nabla \mathbf{u} : \nabla \varphi - (\lambda + \mu) \operatorname{div} \mathbf{u} \operatorname{div} \varphi + p(\rho) \operatorname{div} \varphi + \rho \varphi \cdot \nabla \Phi + \rho f \cdot \varphi dx dt. \end{aligned} \quad (2.5)$$

(v) The equation (1.1) is satisfied in the sense of renormalized solution, i.e.

$$(b(\rho))_t + \operatorname{div}(b(\rho) \mathbf{u}) + (b'(\rho) \rho - b(\rho)) \operatorname{div} \mathbf{u} = 0 \quad (2.6)$$

for any  $b \in C'(\mathbb{R})$  such that  $b'(z) = \text{constant}$ , for any  $z$  large enough, say  $z \geq M$ .

**Lemma 2.1** ([21], Proposition 2.2). *Let  $\gamma > \frac{3}{2}$ . Assume that the initial data  $\rho_0, q_0 = \rho_0 \mathbf{u}_0$  satisfy  $\rho_0 \in L^\gamma(\Omega)$ ,  $\rho_0 \geq 0, q_0 = 0$  whenever  $\rho_0 = 0$ .  $\frac{|q_0|^2}{\rho_0} \in L^1(\Omega)$ . Then the problem (1.1)–(1.4) admits at least one finite energy weak solution  $(\rho, \mathbf{u}, \Phi)$  in  $(0, T) \times \Omega$ .*

We define  $H(s) = \frac{as^\gamma}{\gamma-1}$ . Then the following equalities hold

$$H'(s)s - H(s) = p(s), \quad H''(s)s = p'(s).$$

**Theorem 2.1.** *Suppose that  $f \in L^\infty(0, T; L^1(\Omega) \cap L^\infty(\Omega))$ . Let  $(\rho, \mathbf{u}, \Phi)$  be a finite energy weak solution to the NSP system (1.1)–(1.4) in the sense of Definition 2.1. Let  $r > 0$ ,  $\mathbf{U}, \Psi \in C_0^\infty([0, T] \times \Omega)$  and satisfy*

$$\partial_t r + \operatorname{div}(r \mathbf{U}) = 0, \quad \Delta \Psi = 4\pi g \left( r - \frac{1}{|\Omega|} \int_{\Omega} r dx \right). \quad (2.7)$$

Then the following relative entropy inequality holds for a.e.  $\tau \in (0, T)$ :

$$\begin{aligned} & \varepsilon(\rho, \mathbf{u}, \Phi | r, \mathbf{U}, \Psi)(\tau) + \int_0^\tau \int_{\Omega} \mu |\nabla(\mathbf{u} - \mathbf{U})|^2 + (\lambda + \mu) |\operatorname{div}(\mathbf{u} - \mathbf{U})|^2 dx dt \\ & \leq \varepsilon(\rho_0, \mathbf{u}_0, \Phi_0 | r(0, \cdot), \mathbf{U}(0, \cdot), \Psi(0, \cdot)) + \int_0^\tau \Re(t) dt, \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} \varepsilon(\rho, \mathbf{u}, \Phi | r, \mathbf{U}, \Psi)(\tau) &= \int_{\Omega} \frac{1}{2} \rho |\mathbf{u} - \mathbf{U}|^2(\tau, \cdot) dx + \int_{\Omega} \left( H(\rho) - H'(r)(\rho - r) - H(r) \right)(\tau, \cdot) dx \\ &+ \int_{\Omega} \frac{1}{8\pi g} |\nabla(\Phi - \Psi)|^2(\tau, \cdot) dx, \end{aligned}$$

and the remainder  $\Re(t)$  is defined as

$$\begin{aligned} \Re(t) &= \int_{\Omega} \mu \nabla \mathbf{U} : \nabla(\mathbf{U} - \mathbf{u}) + (\lambda + \mu) \operatorname{div} \mathbf{U} \operatorname{div}(\mathbf{U} - \mathbf{u}) dx \\ &+ \int_{\Omega} \rho (\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla \mathbf{U})(\mathbf{U} - \mathbf{u}) dx + \int_{\Omega} \rho f(\mathbf{u} - \mathbf{U}) dx \\ &+ \int_{\Omega} (r - \rho) \partial_t (H'(r)) + (r \mathbf{U} - \rho \mathbf{u}) \nabla (H'(r)) dx + \int_{\Omega} (p(r) - p(\rho)) \operatorname{div} \mathbf{U} dx \\ &+ \int_{\Omega} \rho (\mathbf{u} - \mathbf{U}) \cdot \nabla \Phi + (r \mathbf{U} - \rho \mathbf{u}) \cdot \nabla (\Phi - \Psi) dx. \end{aligned} \quad (2.9)$$

For convenience, we abbreviate  $\varepsilon(\rho, \mathbf{u}, \Phi|r, \mathbf{U}, \Psi)(t)$  by  $\varepsilon(t)$ .

**Theorem 2.2.** Let  $\gamma > 2$  and  $f \in L^2(0, T; L^{\frac{2\gamma}{\gamma-1}}(\Omega))$ . Suppose that  $(\rho, \mathbf{u}, \Phi)$  is a finite energy weak solution of the NSP system (1.1)–(1.4) in  $(0, T) \times \Omega$  in the sense of Definition 2.1. Assume that  $(r, \mathbf{U}, \Psi)$  is a strong solution of the same problem satisfying

$$0 < \inf_{(0,T) \times \Omega} r \leq r(t, x) \leq \sup_{(0,T) \times \Omega} r < \infty,$$

$$\nabla r \in L^2(0, T; L^q(\Omega)), \quad \nabla^2 \mathbf{U} \in L^2(0, T; L^q(\Omega)), \quad \nabla \Psi \in L^2(0, T; L^q(\Omega))$$

( $q > \max\{3, \frac{2\gamma}{\gamma-1}\}$ ) with the same initial data. Then

$$\rho = r, \quad \mathbf{u} = \mathbf{U}, \quad \Phi = \Psi \quad \text{in } (0, T) \times \Omega.$$

### 3. Relative entropy inequality

#### 3.1. Proof of Theorem 2.1

**Proof.** Taking  $\frac{1}{2}|\mathbf{U}|^2$  as a test function in (2.4), we can get

$$\int_{\Omega} \frac{1}{2} \rho |\mathbf{U}|^2(\tau, \cdot) dx = \int_{\Omega} \frac{1}{2} \rho_0 |\mathbf{U}(0, \cdot)|^2 dx + \int_0^{\tau} \int_{\Omega} \rho \mathbf{U} \cdot \partial_t \mathbf{U} + \rho \mathbf{u} \cdot \nabla \mathbf{U} \cdot \mathbf{U} dx dt. \quad (3.1)$$

Similarly, substituting  $\varphi$  for  $\mathbf{U}$  as a test function, we can obtain

$$\begin{aligned} \int_{\Omega} \rho \mathbf{u} \mathbf{U}(\tau, \cdot) dx &= \int_{\Omega} \rho_0 \mathbf{u}_0 \mathbf{U}(0, \cdot) dx + \int_0^{\tau} \int_{\Omega} \rho \mathbf{u} \partial_t \mathbf{U} + (\rho \mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{U} dx dt \\ &\quad + \int_0^{\tau} \int_{\Omega} -\mu \nabla \mathbf{u} : \nabla \mathbf{U} - (\lambda + \mu) \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{U} + p(\rho) \operatorname{div} \mathbf{U} + \rho \mathbf{U} \cdot \nabla \Phi + \rho f \cdot \mathbf{U} dx dt. \end{aligned} \quad (3.2)$$

By virtue of  $\Phi, \Psi$  satisfying (1.3)(2.7), we can get

$$\left( \frac{1}{8\pi g} \int_{\Omega} |\nabla(\Phi - \Psi)|^2 dx \right)_t = \int_{\Omega} (r \mathbf{U} - \rho \mathbf{u}) \cdot \nabla(\Phi - \Psi) dx. \quad (3.3)$$

From (2.2), we can deduce

$$\begin{aligned} &\int_{\Omega} \frac{1}{2} \rho |\mathbf{u}|^2(\tau, \cdot) + H(\rho)(\tau, \cdot) dx + \int_0^{\tau} \int_{\Omega} \mu |\nabla \mathbf{u}|^2 + (\lambda + \mu) |\operatorname{div} \mathbf{u}|^2 dx dt \\ &\leq \int_{\Omega} \frac{1}{2} \rho |\mathbf{u}_0|^2 + H(\rho_0) dx + \int_0^{\tau} \int_{\Omega} \rho \mathbf{u} \cdot \nabla \Phi + \rho f \cdot \mathbf{u} dx dt. \end{aligned} \quad (3.4)$$

Summing up relations (3.1)–(3.4), we have

$$\begin{aligned} &\int_{\Omega} \left( \frac{1}{2} \rho |\mathbf{u} - \mathbf{U}|^2 + H(\rho) + \frac{1}{8\pi g} |\nabla(\Phi - \Psi)|^2 \right)(\tau, \cdot) dx \\ &+ \int_0^{\tau} \int_{\Omega} \mu \nabla \mathbf{u} : \nabla(\mathbf{u} - \mathbf{U}) + (\lambda + \mu) \operatorname{div} \mathbf{u} \operatorname{div}(\mathbf{u} - \mathbf{U}) dx dt \\ &\leq \int_{\Omega} \frac{1}{2} \rho |\mathbf{u}_0 - \mathbf{U}(0, \cdot)|^2 + H(\rho_0) + \frac{1}{8\pi g} |\nabla(\Phi(\rho_0) - \Psi(\rho_0))|^2 dx + \int_0^{\tau} \int_{\Omega} \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla \Phi + \rho f \cdot (\mathbf{u} - \mathbf{U}) dx dt \\ &+ \int_0^{\tau} \int_{\Omega} \rho(\mathbf{U} - \mathbf{u}) \partial_t \mathbf{U} + \rho \mathbf{u} \cdot \nabla \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) dx dt \\ &+ \int_0^{\tau} \int_{\Omega} (r \mathbf{U} - \rho \mathbf{u}) \cdot \nabla(\Phi - \Psi) dx dt - \int_0^{\tau} \int_{\Omega} p(\rho) \operatorname{div} \mathbf{U} dx dt. \end{aligned} \quad (3.5)$$

Note that

$$\begin{aligned} & \int_{\Omega} \left[ -H(r) - H'(r)(\rho - r) \right] (\tau, \cdot) dx - \int_{\Omega} \left[ -H(r(0, \cdot)) - H'(r(0, \cdot))(\rho_0 - r(0, \cdot)) \right] dx \\ &= \int_0^{\tau} \int_{\Omega} \partial_t(p(r)) - \partial_t(H'(r)\rho) dx dt, \end{aligned}$$

and

$$\int_0^{\tau} \int_{\Omega} r \mathbf{U} \cdot \nabla(H'(r)) + p(r) \operatorname{div} \mathbf{U} dx dt = 0.$$

We may rewrite (3.5) as

$$\begin{aligned} & \varepsilon(\rho, \mathbf{u}, \Phi | r, \mathbf{U}, \Psi)(\tau) + \int_0^{\tau} \int_{\Omega} \mu |\nabla(\mathbf{u} - \mathbf{U})|^2 + (\lambda + \mu) |\operatorname{div}(\mathbf{u} - \mathbf{U})|^2 dx dt \\ & \leq \varepsilon(\rho_0, \mathbf{u}_0, \Phi_0 | r(0, \cdot), \mathbf{U}(0, \cdot), \Psi(0, \cdot)) + \int_0^{\tau} \int_{\Omega} \mu \nabla \mathbf{U} : \nabla(\mathbf{U} - \mathbf{u}) + (\lambda + \mu) \operatorname{div} \mathbf{U} \operatorname{div}(\mathbf{U} - \mathbf{u}) dx \\ & + \int_0^{\tau} \int_{\Omega} \rho (\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla \mathbf{U})(\mathbf{U} - \mathbf{u}) dx + \int_0^{\tau} \int_{\Omega} \rho f(\mathbf{u} - \mathbf{U}) dx \\ & + \int_{\Omega} (r - \rho) \partial_t(H'(r)) + (r \mathbf{U} - \rho \mathbf{u}) \nabla(H'(r)) dx + \int_0^{\tau} \int_{\Omega} (p(r) - p(\rho)) \operatorname{div} \mathbf{U} dx \\ & + \int_0^{\tau} \int_{\Omega} \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla \Phi + (r \mathbf{U} - \rho \mathbf{u}) \cdot \nabla(\Phi - \Psi) dx = \varepsilon(0) + \int_0^{\tau} \mathfrak{R}(t) dt. \end{aligned}$$

The proof of Theorem 2.1 is completed.  $\square$

### 3.2. Extending the admissible class of test function

By means of density argument, we can extend considerably the class of test function  $(r, \mathbf{U}, \Psi)$  appearing in the relative entropy inequality (2.8), (2.9).

For the left hand side of (2.8) to be well defined, the function  $(r, \mathbf{U}, \Psi)$  must belong at least to the class:

$$r \in C_{weak}([0, T]; L^{\gamma}(\Omega)), \mathbf{U} \in C_{weak}([0, T]; L^{\frac{2\gamma}{\gamma-1}}(\Omega)); \quad (3.6)$$

$$\nabla \Psi \in L^{\infty}([0, T]; L^2(\Omega)). \quad (3.7)$$

Similarly, a short inspection of the integrals in (2.9) yields

$$\partial_t \mathbf{U} \in L^1((0, T); L^{\frac{2\gamma}{\gamma-1}}(\Omega)) + L^2((0, T); L^{\frac{6\gamma}{5\gamma-6}}(\Omega)); \quad (3.8)$$

$$\nabla \mathbf{U} \in L^{\infty}((0, T); L^{\frac{3\gamma}{2\gamma-3}}(\Omega)) + L^2((0, T); L^{\frac{6\gamma}{2\gamma-3}}(\Omega)); \quad (3.9)$$

$$\operatorname{div} \mathbf{U} \in L^1((0, T); L^{\infty}(\Omega)); \quad (3.10)$$

$$\nabla^2 \mathbf{U} \in L^1((0, T); L^{\frac{2\gamma}{\gamma-1}}(\Omega)) + L^2((0, T); L^{\frac{6\gamma}{5\gamma-6}}(\Omega)); \quad (3.11)$$

$$\nabla \Psi \in L^1((0, T); L^{\frac{2\gamma}{\gamma-1}}(\Omega)) + L^2((0, T); L^{\frac{6\gamma}{5\gamma-6}}(\Omega)). \quad (3.12)$$

Moreover, the function  $r$  must be bounded away from zero, and

$$\partial_t(H'(r)) \in L^1((0, T); L^{\frac{\gamma}{\gamma-1}}(\Omega)); \quad (3.13)$$

$$\nabla(H'(r)) \in L^2((0, T); L^{\frac{6\gamma}{5\gamma-6}}(\Omega)) + L^1((0, T); L^{\frac{2\gamma}{\gamma-1}}(\Omega)). \quad (3.14)$$

It is easy to prove that the relative entropy inequality (2.8)(2.9) can be extended to  $(r, \mathbf{U}, \Psi)$  satisfying (3.6)–(3.14) by density argument.

#### 4. Weak-strong uniqueness

In order to prove [Theorem 2.2](#), we firstly rewrite the expression of  $\mathfrak{R}(t)$ .

**Lemma 4.1.** For  $r > 0$ , the remainder  $\mathfrak{R}(t)$  can read as

$$\begin{aligned}\mathfrak{R}(t) &= \int_{\Omega} \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) dx \\ &\quad + \int_{\Omega} \mu r^{-1}(\rho - r) \Delta \mathbf{U}(\mathbf{U} - \mathbf{u}) + (\mu + \lambda) r^{-1}(\rho - r)(\mathbf{U} - \mathbf{u}) \cdot \nabla \operatorname{div} \mathbf{U} dx \\ &\quad + \int_{\Omega} (r - \rho) \mathbf{U} \cdot \nabla(\Phi - \Psi) dx + \int_{\Omega} \left( p(r) - p'(r)(r - \rho) - p(\rho) \right) \operatorname{div} \mathbf{U} dx.\end{aligned}\quad (4.1)$$

**Proof.** For  $r > 0$ , there exists  $a_0 > 0$  so that  $0 < a_0 \leq r < \infty$ . Since  $(r, \mathbf{U}, \Psi)$  is the strong solution of the system [\(1.1\)–\(1.3\)](#), the following equality holds

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} = \mu r^{-1} \Delta \mathbf{U} + (\lambda + \mu) r^{-1} \nabla \operatorname{div} \mathbf{U} - r^{-1} \nabla p(r) + \nabla \Psi + f. \quad (4.2)$$

Substituting [\(4.2\)](#) into [\(2.9\)](#), one has

$$\begin{aligned}\mathfrak{R}(t) &= \int_{\Omega} \mu \nabla \mathbf{U} : \nabla(\mathbf{U} - \mathbf{u}) + (\lambda + \mu) \operatorname{div} \mathbf{U} \operatorname{div}(\mathbf{U} - \mathbf{u}) dx \\ &\quad + \int_{\Omega} \mu \rho r^{-1} \Delta \mathbf{U}(\mathbf{U} - \mathbf{u}) + (\mu + \lambda) \rho r^{-1} \nabla \operatorname{div} \mathbf{U}(\mathbf{U} - \mathbf{u}) dx \\ &\quad + \int_{\Omega} \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) dx + \int_{\Omega} \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla(H'(r)) dx \\ &\quad + \int_{\Omega} (r - \rho) \partial_t(H'(r)) + (r\mathbf{U} - \rho\mathbf{u}) \nabla(H'(r)) dx + \int_{\Omega} \left( p(r) - p(\rho) \right) \operatorname{div} \mathbf{U} dx \\ &\quad + \int_{\Omega} \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla(\Phi - \Psi) + (r\mathbf{U} - \rho\mathbf{u}) \cdot \nabla(\Phi - \Psi) dx.\end{aligned}\quad (4.3)$$

Note that

$$(r - \rho) \partial_t(H'(r)) + (r\mathbf{U} - \rho\mathbf{u}) \nabla(H'(r)) + \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla(H'(r)) = -(r - \rho) p'(r) \operatorname{div} \mathbf{U}.$$

Then we can rewrite  $\mathfrak{R}(t)$  as [\(4.1\)](#). The proof of [Lemma 4.1](#) is completed.  $\square$

In order to estimate the remainder  $\mathfrak{R}(t)$ , we can deduce the following lemma:

**Lemma 4.2.** Let  $\rho \geq 0$  and  $0 < a_0 \leq r \leq b_0 < \infty$ . There exist  $a_1 \in (0, a_0)$ ,  $M \gg 1$  and a constant  $c > 0$  so that

$$H(\rho) - H'(r)(\rho - r) - H(r) \geq \begin{cases} c(\rho - r)^2 & \text{if } a_1 \leq \rho \leq Mr; \\ \frac{p(r)}{2} & \text{if } 0 \leq \rho < a_1; \\ c\rho^\gamma & \text{if } \rho > Mr. \end{cases} \quad (4.4)$$

**Proof.** By virtue of Taylor's formula and the definition of  $H(r)$ , it is easy to infer the inequality [\(4.4\)](#). Hence, we omit to prove this lemma.  $\square$

Finally, we prove [Theorem 2.2](#).

**Proof. Step1:** we estimate the remainder  $\mathfrak{R}(t)$ .

For  $\nabla \mathbf{U}$ ,  $\operatorname{div} \mathbf{U} \in L^1(0, T; L^\infty(\Omega))$ , it is easy to infer that

$$I_1 = \int_{\Omega} \left( \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) - \left( p(\rho) - p'(r)(\rho - r) - p(r) \right) \operatorname{div} \mathbf{U} \right) dx \leq \|\nabla \mathbf{U}\|_{L^\infty(\Omega)} \varepsilon(t) \leq \eta(t) \varepsilon(t). \quad (4.5)$$

The second term of  $\mathfrak{R}(t)$  can be rewritten as

$$\begin{aligned} I_2 &= \int_{\Omega} \mu r^{-1}(\rho - r) \Delta \mathbf{U}(\mathbf{U} - \mathbf{u}) + (\mu + \lambda) r^{-1}(\rho - r)(\mathbf{U} - \mathbf{u}) \cdot \nabla \operatorname{div} \mathbf{U} dx \\ &= \left( \int_{\{a_1 \leq \rho \leq Mr\}} + \int_{\{0 \leq \rho < a_1\}} + \int_{\{\rho > Mr\}} \right) r^{-1}(\rho - r) \left( \mu \Delta \mathbf{U}(\mathbf{U} - \mathbf{u}) + (\mu + \lambda)(\mathbf{U} - \mathbf{u}) \cdot \nabla \operatorname{div} \mathbf{U} \right) dx \\ &:= I_{21} + I_{22} + I_{23} \end{aligned}$$

By virtue of Hölder's inequality, Sobolev's inequality and [Lemma 4.2](#), one has

$$\begin{aligned} I_{21} &\leq C \|\nabla^2 \mathbf{U}\|_{L^3(\Omega)} \|\rho - r\|_{L^2(\{a_1 \leq \rho \leq Mr\})} \|\mathbf{U} - \mathbf{u}\|_{L^6(\Omega)} \\ &\leq C \|\nabla^2 \mathbf{U}\|_{L^3(\Omega)}^2 \int_{\{a_1 \leq \rho \leq Mr\}} (\rho - r)^2 dx + \delta \int_{\Omega} |\nabla(\mathbf{U} - \mathbf{u})|^2 dx \\ &\leq \eta(t) \varepsilon(t) + \delta \int_{\Omega} |\nabla(\mathbf{U} - \mathbf{u})|^2 dx \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} I_{22} &\leq \|\nabla^2 \mathbf{U}\|_{L^3(\Omega)} \left( \int_{\{0 \leq \rho < a_1\}} 1 dx \right)^{\frac{1}{2}} \|\mathbf{U} - \mathbf{u}\|_{L^6(\Omega)} \\ &\leq C \|\nabla^2 \mathbf{U}\|_{L^3(\Omega)}^2 \int_{\{0 \leq \rho < a_1\}} H(\rho) - H'(r)(\rho - r) - H(r) dx + \delta \int_{\Omega} |\nabla(\mathbf{U} - \mathbf{u})|^2 dx \\ &\leq \eta(t) \varepsilon(t) + \delta \int_{\Omega} |\nabla(\mathbf{U} - \mathbf{u})|^2 dx. \end{aligned} \quad (4.7)$$

with  $\delta > 0$  sufficient small.

From [Lemma 4.2](#), we can get

$$\|\rho\|_{L^\gamma(\{\rho > Mr\})} \leq c[\varepsilon(t)]^{\frac{1}{\gamma}} \quad \text{and} \quad \|\rho^{\frac{\gamma}{2}}\|_{L^2(\{\rho > Mr\})} \leq c[\varepsilon(t)]^{\frac{1}{2}}. \quad (4.8)$$

Due to  $\gamma > 2$  and by using Sobolev's inequality, Hölder's inequality and [\(4.8\)](#), we can deduce

$$\begin{aligned} I_{23} &\leq C \int_{\{\rho > Mr\}} \left| \frac{\rho - r}{\rho r} \right| \rho^{\frac{\gamma}{2}} |\mathbf{U} - \mathbf{u}| |\nabla^2 \mathbf{U}| dx \leq C \|\nabla^2 \mathbf{U}\|_{L^3(\Omega)} \|\rho^{\frac{\gamma}{2}}\|_{L^2(\{\rho > Mr\})} \|\mathbf{U} - \mathbf{u}\|_{L^6(\Omega)} \\ &\leq C \|\nabla^2 \mathbf{U}\|_{L^3(\Omega)}^2 \varepsilon(t) + \delta \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2(\Omega)}^2 \\ &\leq \eta(t) \varepsilon(t) + \delta \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.9)$$

The third term of  $\mathfrak{R}(t)$  is denoted as

$$\begin{aligned} I_3 &= \int_{\Omega} (r - \rho) \mathbf{U} \cdot \nabla(\Phi - \Psi) dx = \left( \int_{\{a_1 \leq \rho \leq Mr\}} + \int_{\{0 \leq \rho < a_1\}} + \int_{\{\rho > Mr\}} \right) (r - \rho) \mathbf{U} \cdot \nabla(\Phi - \Psi) dx \\ &:= I_{31} + I_{32} + I_{33}. \end{aligned}$$

By [Lemma 4.2](#) and Hölder's inequality, we can obtain

$$I_{31} \leq C \|\mathbf{U}\|_{L^\infty(\Omega)} \left( \int_{\{a_1 \leq \rho \leq Mr\}} (r - \rho)^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla(\Phi - \Psi)|^2 dx \right)^{\frac{1}{2}} \leq \eta(t) \varepsilon(t). \quad (4.10)$$

and

$$\begin{aligned} I_{32} &\leq C \|\mathbf{U}\|_{L^\infty(\Omega)} \left( \int_{\{0 \leq \rho < a_1\}} 1 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla(\Phi - \Psi)|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \|\mathbf{U}\|_{L^\infty(\Omega)} \left( \int_{\{0 \leq \rho < a_1\}} H(\rho) - H'(r)(\rho - r) - H(r) dx \right)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}(t) \\ &\leq \eta(t) \varepsilon(t). \end{aligned} \quad (4.11)$$

In view of (4.8) and Lemma 4.2, we have

$$\begin{aligned} I_{33} &\leq C \int_{\{\rho > Mr\}} \rho^{\frac{\gamma}{2}} |\mathbf{U}| |\nabla(\Phi - \Psi)| dx \\ &\leq C \|\mathbf{U}\|_{L^\infty(\Omega)} \|\rho^{\frac{\gamma}{2}}\|_{L^2(\{\rho > Mr\})} \|\nabla(\Phi - \Psi)\|_{L^2(\Omega)} \leq C \|\mathbf{U}\|_{L^\infty(\Omega)} \varepsilon(t) \\ &\leq \eta(t) \varepsilon(t) \end{aligned} \quad (4.12)$$

with  $\gamma > 2$ .

**Step2:** Basing on the estimate of  $\mathfrak{R}(t)$ , we prove the result of Theorem 2.2.

Plugging relations (4.5)–(4.12) to (2.8), we can deduce

$$\varepsilon(\tau) + \int_0^\tau \int_\Omega \mu |\nabla(\mathbf{u} - \mathbf{U})|^2 + (\lambda + \mu) |\operatorname{div}(\mathbf{u} - \mathbf{U})|^2 dx dt \leq \int_0^\tau \eta(t) \varepsilon(t) dt$$

with  $\eta(t) \in L^1(0, T)$ . By using Gronwall's inequality, we infer  $\varepsilon(t) = 0$  in  $(0, T)$ . This implies  $\rho = r$ ,  $\mathbf{u} = \mathbf{U}$ ,  $\Phi = \Psi$ . The proof of Theorem 2.2 is completed.  $\square$

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